Math 131B-2: Homework 6

Due: May 12, 2014

1. Read Apostol Sections 9.6, 9.8, 9.10, 9.14-15.

2. Do problems 9.2, 9.3, 9.14, 9.16, 9.22 in Apostol. [For several of these problems, it will be extremely helpful to keep in mind Question 6 from last week].

3. *Dini's Theorem* Let $f_n \colon X \to \mathbb{R}$ be a sequence of continuous functions on a compact metric space X which converges pointwise to a continuous function $f \colon X \to \mathbb{R}$ and suppose that for each x the sequence $\{f_n(x)\}$ is increasing, i.e. $f_n(x) \leq f_m(x)$ for all $n < m$. We will prove that $\{f_n\}$ in fact converges to f uniformly.

   - Let $\epsilon > 0$. For each $n \in \mathbb{N}$, let $g_n = f - f_n(x)$ and show that $\{x \in X : |g_n(x)| < \epsilon\}$ is an open set V_n^ϵ of X. Moreover, show that for $n < m$, we have the inclusion $V_n^\epsilon \subseteq V_m^\epsilon$.
   - Show that $X \subset V_N^\epsilon$ for some $N > 0$. [Hint: the V_N^ϵ cover X.]
   - Prove that $f_n \to f$ uniformly.
   - Give an example showing that the theorem is not true if we do not require f to be continuous. [Hint: Consider the examples done in lecture.]

   This is one of very few situations where pointwise convergence implies uniform convergence.

4. *A question of arc length.* Recall that the sequence $f_n(x) = \frac{1}{n}\sin(nx)$ converges uniformly to $f(x) = 0$ on the real line. Moreover, recall, e.g. from your calculus class, that whenever g is a continuous function on $[a, b]$ which is differentiable on (a, b) with continuous derivative g', the arclength of the curve $\{(x, g(x)) \in \mathbb{R}^2 : x \in [a, b]\}$ is $S_a^b(f) = \int_a^b \sqrt{1 + f'(x)^2}dx$. Show that $S_0^\pi(f_n)$ does not converge to $S_0^\pi(f)$. [Hint: You can't actually do the integrals you get, you're looking for a lower bound which is greater than π.]

   Ergo it is possible for a sequence of functions to converge uniformly on an interval without convergence of the arc lengths of their graphs over the interval. What kind of hypothesis do you think you would need to add to get convergent arc lengths?

5. *Continuity makes life easier.* In class we proved that if $f_n \colon (a, b) \to \mathbb{R}$ is a sequence of differentiable functions such that the derivatives f_n' converge uniformly to some g on (a, b) and $\{f_n(x_0)\}$ converges for at least one x_0 in (a, b), then there is a differentiable function f such that $f_n \to f$ uniformly, and $f'(x) = g(x)$. If we're willing to add the assumption that each of the f_n' is continuous on (a, b), we can give an easier proof.

   - Show that $\int_{x_0}^x f_n'$ and $\int_{x_0}^x g$ both exist for every $x \in (a, b)$, and the sequence of functions $\int_{x_0}^x f_n'$ converges uniformly to $\int_{x_0}^x g$. [Hint: This is extremely straightforward.]
   - Observe that by FTC Part I, $\int_{x_0}^x f_n' = f_n(x) - f_n(x_0)$. Therefore $h_n(x) = f_n(x) - f_n(x_0)$ converges uniformly to $\int_{x_0}^x g$.
   - Let $L = \lim_{n\to\infty} f_n(x_0)$. Consider the function $f \colon (a, b) \to \mathbb{R}$ defined by $f(x) = L + \int_{x_0}^x g$. Prove, using the second part of this problem, that $f_n(x) \to f(x)$ uniformly.
   - Prove, using FTC Part II, that $f'(x) = g(x)$ on (a, b). [This is the only place where it is important that the f_n' are continuous, and not merely integrable.]